CONDITIONS NECESSARY AND SUFFICIENT FOR THE EXISTENCE OF A STIELTJES INTEGRAL

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The purpose of this note is to suggest a method for deriving a necessary and sufficient condition for the existence of the Stieltjes integral of $f(x)$ as to $u(x)$ from a to b in each of several forms generalizing those frequently employed (see *Encyclopédie des sciences mathématiques*, II_1, pp. 171–174) in the special case of Cauchy-Riemann integration—where $u(x)$ is of bounded variation and $f(x)$ is bounded on the interval (ab), $M \geqq f(x) \geqq m$. Bliss (Proceedings **3**, 1917, pp. 633–637) has obtained one of these forms, perhaps the most satisfying of any; this note closes with his theorem, of which we give a new demonstration. Some of the other theorems stated (though here derived otherwise) are immediate consequences of the one due to Bliss or are otherwise intimately related to it, as the reader will readily see. Since it is believed that the present treatment will be found useful in connection with that of Bliss, our notation has been made to conform to his; moreover, reference to his paper is given for such isolated steps in the proof as may readily be supplied from it.

For a given partition π of (ab) due to the points $x_0 = a$, $x_1, x_2, \ldots,$ x_{n-1}, $x_n = b$, $0 < x_i - x_{i-1} < \delta$, form the sums

$$s_\pi = \sum_{i=1}^{n} M_i \Delta_i u, \quad S_\pi = \sum_{i=1}^{n} f(X_i)\,\Delta_i u, \quad \sigma_\pi = \sum_{i=1}^{n} m_i \Delta_i u,$$

where $\Delta_i u = u(x_i) - u(x_{i-1})$, X_i is any point of the interval (x_{i-1}, x_i), and $M_i[m_i]$ is the least upper bound [the greatest lower bound] of $f(x)$ on (x_{i-1}, x_i). If the limit of S_π exists as δ approaches zero, this limit is the Stieltjes integral of $f(x)$ as to $u(x)$ from a to b. In case $u(x)$ is a monotonic non-decreasing function, the limit, if existing, of $s_\pi[\sigma_\pi]$, as δ approaches zero, is here called the upper [lower] Stieltjes integral of $f(x)$ as to $u(x)$ from a to b.

Let us determine conditions under which the upper integral shall certainly exist when $u(x)$ is monotonic non-decreasing. If we form a repartition π' of (ab) as to π by taking the points forming π and certain additional points, it is clear that $S_{\pi'} \leqq S_\pi$. Moreover, there is ob-

viously a lower bound to $S_{\pi'}$. Hence if the number of divisions of (ab) is increased by repartitions in such wise that δ approaches zero, the sum $S_{\pi'}$ approaches a definite finite limit. Let us ask under what conditions we shall be led to a contradiction by supposing that two convergent sequences of sums σ_π for sequences of partitions π (whether formed by repartitions or not) with norms δ approaching 0 lead to different limits N_1 and N_2 where $N_1 < N_2$. Let ξ and η be two arbitrarily small positive numbers such that $N_1 + \xi + \eta < N_2$. Let π_1 be a partition of (ab) into s intervals belonging to the sequence by which N_1 is defined and let s be so great that $S_{\pi_1} < N_1 + \xi$. Let π_2 be a partition of (ab) into t intervals where t is an integer greater than s and subject to being made as large as one pleases. Let π_3 be a partition formed by the points of π_1 and π_2, so that π_3 is a repartition of both π_1 and π_2. Then we have

$$S_{\pi_2} - S_{\pi_3} \leqq \sum_{\alpha=1}^{s-k} (M_{i_\alpha} - m_{i_\alpha})\, \Delta_{i_\alpha} u,$$

where the intervals $(x_{i_\alpha-1}, x_{i_\alpha})$, obviously at most $s - 1$ in number, are all the intervals of π_2 which are separated into parts in forming π_3. Since t may be made large at our choice and since the sum in the second member of the foregoing relation never has more than $s - 1$ terms, whatever the value of t, it is clear that t may be chosen so large that this sum is less than η provided that $u(x)$ is continuous at the discontinuities of $f(x)$. Then we have $S_{\pi_2} \leqq S_{\pi_3} + \eta$. But we have seen that $S_{\pi_3} \leqq S_{\pi_1} < N_1 + \xi$. Hence we have $S_{\pi_2} < N_1 + \xi + \eta$. But, as t increases, S_{π_2} approaches N_2. Hence we have $N_2 \leqq N_1 + \xi + \eta$, contrary to the hypothesis $N_1 + \xi + \eta < N_2$. Hence the upper integral of $f(x)$ as to a monotonic non-decreasing function $u(x)$ exists provided that $u(x)$ is continuous at the points of discontinuity of $f(x)$. The corresponding result may likewise be proved for the case of a non-decreasing function $u(x)$; and also for the case of the lower integral. Hence, since every function of bounded variation $u(x)$ may be expressed as the difference of two monotonic non-decreasing functions which are continuous at the points of continuity of $u(x)$, we have the following theorem:

THEOREM I. *If $u(x)$ is of bounded variation and $f(x)$ is bounded on (ab) and if $f(x)$ is continuous at the points of discontinuity of $u(x)$, then the limits as δ approaches 0 of the sums s_π, σ_π of the preceding paragraph both exist.*

Let us now write the function $u(x)$ in the form

$$u(x) = u(a) + P(x) - N(x),$$

where $P(x)$ and $N(x)$ are respectively the positive and the negative variation of $u(x)$ on (ax); and by $U(x)$ denote the total variation $P(x) + N(x)$. Then $P(x)$, $N(x)$, $U(x)$ are continuous at the points of continuity of $u(x)$.

A sufficient condition for the existence of the integral of $f(x)$ as to $u(x)$ from a to b is the existence of the integral of $f(x)$ as to $P(x)$ and as to $N(x)$. Sufficient to this is the existence and equality of the upper and lower integrals of $f(x)$ as to $P(x)$ and as to $N(x)$. In case $u(x)$ is continuous or $f(x)$ is continuous at the discontinuities of $u(x)$, these upper and lower integrals surely exist and a sufficient condition for their equality is that the sums

$$\sum_{i=1}^{n}(M_i-m_i)\{P(x_i)-P(x_{i-1})\} \text{ and } \sum_{i=1}^{n}(M_i-m_i)\{N(x_i)-N(x_{i-1})\}$$

shall have the limit zero as δ approaches zero. Hence a sufficient condition for the existence of the integral of $f(x)$ as to $u(x)$ from a to b is that

$$\lim_{\delta=0}\sum_{i=1}^{n}(M_i-m_i)\{U(x_i)-U(x_{i-1})\} = 0. \qquad (1)$$

Bliss (l. c., p. 634, ll. 1–10) has shown that a necessary condition for the existence of the integral is that

$$\lim_{\delta=0}\sum_{i=1}^{n}(M_i-m_i)\,|\,u(x_i)-u(x_{i-1})\,| = 0. \qquad (2)$$

From this necessary condition if follows readily that $f(x)$ must be continuous at the points of discontinuity of $u(x)$ (see Bliss, l. c., p. 636, ll. 7–17). If we write $u(x) = v(x) + j(x)$ (Bliss, p. 636, ll. 1–7), where $j(x)$ is the function of 'jumps' of $u(x)$, it may be shown (Bliss, p. 636, ll. 18–36) that the integral of $f(x)$ as to $j(x)$ exists whenever that as to $u(x)$ exists. In the same way it may be shown that the integral of $f(x)$ as to $J(x)$ must also exist, where $J(x)$ is the total variation of $j(x)$ on the interval (ax). Hence a necessary condition for the existence of the integral of $f(x)$ as to $u(x)$ is the existence of the integral of $f(x)$ as to $v(x)$.

We propose to show next that the existence of the integral of $f(x)$ as to $u(x)$ implies that of $f(x)$ as to $U(x)$. In view of the results of the

preceding paragraph and of the fact that $U(x)$ is obviously equal to $V(x) + J(x)$, where $V(x)$ is the total variation of $v(x)$ on (ax), it is obviously sufficient to prove this for the case when $u(x)$ is a continuous function. Now, when $u(x)$ is continuous, we have (Vallée Poussin, *Cours d'Analyse*, vol. 1, 3rd edn, p. 73)

$$\lim_{\delta=0} \sum_{i=1}^{n} | u(x_i) - u(x_{i-1}) | = U(b), \tag{3}$$

with a like relation when b is replaced by x and the interval (ab) by the interval (ax). Since no $| u(x_i) - u(x_{i-1}) |$ is greater than the corresponding difference $U(x_i) - U(x_{i-1})$ and since the sum of the latter differences, for $i = 1, 2, \ldots, n$, is $U(b)$, we see that

$$\lim_{\delta=0} \sum_{i=1}^{n} [U(x_i) - U(x_{i-1}) - | u(x_i) - u(x_{i-1}) |] = 0 \tag{4}$$

and that no bracketed term here is negative. Hence for every ϵ there exists a δ_1 such that the i^{th} term ($i = 1, 2, \ldots, n$) of the sum in (4) is less than ϵ when $\delta < \delta_1$. Hence, for such δ, we have

$$0 \leqq \sum_{i=1}^{n} (M_i - m_i)[U(x_i) - U(x_{i-1}) - | u(x_i) - u(x_{i-1}) |] < (M - m)\epsilon \tag{5}$$

Hence, as δ approaches zero the sum in (5) approaches zero as a limit. This result and relation (2) imply (1). But the latter is sufficient to the existence of the integral of $f(x)$ as to $U(x)$, and indeed as to $u(x)$.

We are thus led to the following theorem:

THEOREM II. *If $u(x)$ is of bounded variation and $f(x)$ is bounded on (ab), then a necessary and sufficient condition for the existence of the integral of $f(x)$ as to $u(x)$ is the existence of the integral of $f(x)$ as to $U(x)$.*

Since (1) and (2) are identical when $u(x) = U(x)$ we now have readily the following theorems:

THEOREM III. *A necessary and sufficient condition for the existence of the integral of the bounded function $f(x)$ as to the function $u(x)$ of bounded variation is that the upper and lower integrals of $f(x)$ as to the total variation function $U(x)$ of $u(x)$ shall exist and be equal.*

THEOREM IV. *A necessary and sufficient condition for the existence of the integral from a to b of the bounded function $f(x)$ as to the function $u(x)$ of bounded variation is that the total oscillation of $f(x)$ as to $u(x)$ from a to b shall be zero, that is, that*

$$\lim_{\delta=0} \sum_{i=1}^{n} (M_i - m_i)[U(x_i) - U(x_{i-1})] = 0.$$

THEOREM V. *In order that a bounded function $f(x)$ shall be integrable from a to b as to a function $u(x)$ of bounded variation, it is necessary and sufficient that the interval (ab) may be divided into partial intervals so that the total variation of $u(x)$ in those in which the oscillation of $f(x)$ is greater than an arbitrarily preassigned positive number ω shall also be as small as one wishes.*

THEOREM VI. *If $u(x)$ is of bounded variation and $f(x)$ is bounded on the interval (ab), then a necessary and sufficient condition for the existence of the integral of $f(x)$ as to $u(x)$ from a to b is that the total variation of $u(x)$ on the set D of discontinuities of $f(x)$ shall be zero (Theorem of Bliss).*

Of the four preceding theorems the only one needing further proof is the last. [For the definition of the total variation of $u(x)$ on a set of points, see Bliss, l. c., p. 633, ll. 12–19.] Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of positive numbers decreasing monotonically toward zero, and let $D_1, D_2, D_3, \ldots$ be the closed set of points at which the oscillation of $f(x)$ is $\geqq \epsilon_1, \geqq \epsilon_2, \geqq \epsilon_3, \ldots$ Then the set D of discontinuities of $f(x)$ is the limit of the set D_n when n is indefinitely increased. Now if $f(x)$ is integrable as to $u(x)$ we have seen that the interval (ab) may be divided into partial intervals so that the total variation of $u(x)$ on those in which the oscillation of $f(x)$ is greater than an arbitrarily preassigned positive number shall be as small as one pleases; and this implies that the total variation of $u(x)$ on D_n, and hence on D, is zero. Again, if the total variation of $u(x)$ on D is zero so is it on D_n for every n; and hence $f(x)$ is integrable as to $u(x)$ since it is such that the interval (ab) may be divided into partial intervals so that the sum of those in which the oscillation is greater than an arbitrarily preassigned positive number shall be as small as one pleases.

TRANSFORMATIONS OF CYCLIC SYSTEMS OF CIRCLES

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When two surfaces, S and $\bar{S}$, are applicable, there is a unique conjugate system on S which corresponds to a conjugate system on $\bar{S}$. Denote these conjugate systems, or *nets*, by N and $\bar{N}$ respectively.